Simple Groups of Lie Type

↳ how Lie theory can be used to construct finite simple groups through the work of Chevalley and creation of Chevalley basis → important because these Lie type groups are a major part of the classification of all finite simple groups.

Chevalley Basis 4.1

~similar idea of going from a root system to Lie algebra

Let g be a non-trivial simple Lie algebra with Cartan subalgebra h

Cartan decomposition: $$g = h \oplus \sum_{\alpha \in \Phi} g_\alpha$$

direct sum of h and root spaces

Φ root system

g_α root space

construct a basis:

- choose fundamental coroots $h_1, \dots, h_\ell \in h$

  where $\ell = \dim h$ or # of simple roots

  as any coroot $h_\alpha \in h$ corresponding to any root $\alpha \in \Phi$ is a linear combination of fundamental coroots $h_1, \dots, h_\ell$ with coefficients in $\mathbb{Z}$

- For each root α, choose nonzero vectors $e_\alpha \in g_\alpha$

since $\dim g_\alpha = 1$, e_α is determined by a scalar

(can show) the lie bracket $[e_\alpha, e_{-\alpha}]$ is a nonzero multiple of h_α

- so we choose $e_\alpha \in g_\alpha$, $e_{-\alpha} \in g_{-\alpha}$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$

consider $[e_\alpha e_\beta]$ where $\alpha + \beta \neq 0$

then for $x \in h$,

$$[x[e_\alpha, e_\beta]] = [[x e_\alpha] e_\beta] + [e_\alpha [x e_\beta]]$$
$$= \alpha(x)[e_\alpha e_\beta] + \beta(x)[e_\alpha e_\beta]$$
$$= (\alpha + \beta)(x)[e_\alpha e_\beta]$$

so if $\alpha + \beta \notin \Phi$, $[e_\alpha e_\beta] = 0$

if $\alpha + \beta \in \Phi$, $[e_\alpha e_\beta] \in g_{\alpha+\beta}$

suppose $\alpha+\beta \in \Phi$ and let $[e_\alpha e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$

integer structure constant
ie coefficients in the formula for $[x_i, x_j]$

then $[e_{-\alpha} e_{-\beta}] = N_{-\alpha,-\beta} e_{-\alpha-\beta}$

$$N_{\alpha,\beta} N_{-\alpha,-\beta} = -(p+1)^2$$

where p is the

non negative integer s.t. $\beta, -\alpha+\beta, -2\alpha+\beta, \dots,$

$-p\alpha+\beta$ are roots but $-(p+1)\alpha+\beta$ is not a

root

then, for vectors $e_\alpha \in g_\alpha$ can write

$$[e_\alpha e_\beta] = \pm(p+1) e_{\alpha+\beta} \text{ when } \alpha+\beta \in \Phi$$

<u>multiplication of basis elements</u> are given by

$[h_i \; h_j] = 0$

$$[h_i \; e_\alpha] = \frac{2\langle \alpha_i, \alpha \rangle}{\langle \alpha_i, \alpha_i \rangle} e_\alpha$$

$[e_\alpha e_{-\alpha}] = h_\alpha$, a $\mathbb{Z}$-combination of $h_1, \dots, h_\ell$

$[e_\alpha e_\beta] = \pm(p+1) e_{\alpha+\beta}$ if $\alpha+\beta \in \Phi$

$[e_\alpha e_\beta] = 0$ if $\alpha+\beta \notin \Phi$, $\alpha+\beta \neq 0$

Thus, the Lie product of any two basis elements is a $\mathbb{Z}$-combination of basis elements

This kind of basis is called a Chevalley basis

<u>Automorphisms</u>

For any $\alpha \in \Phi$, $\lambda \in \mathbb{C}$ consider the map

$$\operatorname{ad}(\lambda e_\alpha) : g \to g$$

given by $\operatorname{ad}(\lambda e_\alpha) x = \lambda [e_\alpha x]$

linear map $\exp \operatorname{ad}(\lambda e_\alpha) = 1 + \operatorname{ad}(\lambda e_\alpha) + \frac{\operatorname{ad}(\lambda e_\alpha)^2}{2!} + \dots$

because
this map is nilpotent as $(\operatorname{ad}(\lambda e_\alpha))^k = 0$ for some k
the sum is finite

Then, $\exp \operatorname{ad}(\lambda e_\alpha)$ transforms every element of the Chevalley basis into a linear combination of basis elements with coefficients which are polynomials in λ with coefficients in $\mathbb{Z}$

The elements $\exp \operatorname{ad}(\lambda e_\alpha)$ form the adjoint algebraic group with Lie algebra g.

Chevalley Groups   4.2 4.3

def $G_{ad}(k)$ as the adjoint Chevalley group over a field k built from matrices representing $\exp(\mathrm{ad}(A e_\alpha))$ where $A \in k$

$G_{ad}(k)$ is simple, apart from a small finite number of exceptions when k is finite

examples:

$A_\ell \to PSL_{\ell+1}(k)$ projective special linear group of degree $\ell+1$ over k

$C_\ell \to PSp_{2\ell}(k)$

projective symplectic group of degree 2ℓ over k

(ℓ is rank of root system)

Finite Chevalley Groups

consider k is a finite field

# of elements in any finite field is a prime power and for each prime power $q = p^e$ there is just one field F_q with q elements so when $k = F_q$ we write

$G_{ad}(k) = G_{ad}(q)$

order formula $|G_{ad}(q)| = \frac{1}{d} q^{|\Phi^+|}(q^{d_1}-1)(q^{d_2}-1)\dots(q^{d_\ell}-1)$

explain if have time

$d_1, \dots, d_\ell$ are degrees of basic polynomial invariants (Weyl group)

The group $G_{ad}(q)$ is a finite simple group except in case $A_1(2)$, $A_1(3)$, $B_2(2)$, $G_2(2)$

These groups are called the simple Chevalley groups.

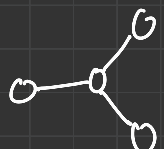
<u>Twisted Groups</u> - also finite simple groups obtainable from Lie theory

twisted groups are obtained whenever the Dynkin diagram has a symmetry

ex.

A_ℓ  $\ell \geq 2$

D_ℓ  $\ell \geq 4$

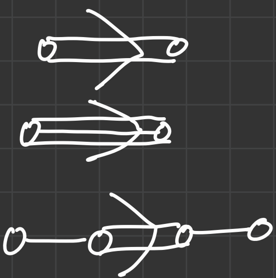
D_4 

Suzuki and Ree groups

arise from symmetries of Dynkin diagrams that don't preserve root lengths

so they have symmetry without arrows, but not when arrows are included

ex: B_2, G_2, F_4



Chevalley groups, twisted groups, Suzuki and Ree groups over finite fields are called the finite groups of Lie type.

Classification of finite simple groups

completed in 1981

every finite simple group is isomorphic to one of:

- cyclic group of prime order
- alternating group of degree $n \geq 5$
- finite simple group of Lie type
- one of 26 sporadic simple group

note:

- most finite simple groups are of Lie type
- the monster group is the largest sporadic group and it has links to Lie theory through Kac-Moody algebras and string theory

Recap / Overview

- Chevalley basis lets us define Lie algebras over arbitrary fields

  ↓

  this gives rise to finite Chevalley groups and the other groups like the twisted group

  ↓

  these groups make up the majority of the finite simple group classification